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New compact equation for numerical simulation of freak waves on deep water

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Abstract. Considering surface gravity waves which propagate in same direction we applied canonical transformation to a water wave equation and drastically simplify the Hamiltonian. After this transformation, corresponding equation of motion is written in x-space in a compact form. This new equation is suitable for analytical studies and numerical simulations. Localized in space breather-type solution was found numerically by using iterative Petviashvili method. Numerical simulation of breathers collision shows the stability of such solutions. We observed the freak wave formation in numerical simulations of sea surface waving in the framework of new equation.

1. Compact equation

A one dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field is considered. The system is Hamiltonian system and Hamiltonian variables are the free surface elevation $\eta(x, t)$ and the velocity potential at the free boundary $\psi(x, t)$ [1]. Considering not very steep surface waves (small parameter is a steepness of the waves $\mu \sim \eta'_x$), the Hamiltonian can be represented by an infinite power series expansion on its natural variables. The Hamiltonian expansion up to 4th order as follows:

$$H = \frac{1}{2} \int g\eta^2 + \psi \hat{k} \psi dx - \frac{1}{2} \int \{(\hat{k}\psi)^2 - (\psi_x)^2\} \eta dx + \frac{1}{2} \int \{\psi_{xx} \eta^2 \hat{k} \psi + \psi \hat{k} (\eta \hat{k} (\eta \hat{k} \psi))\} dx \quad (1)$$

Here \hat{k} is modulus operator ($|k|$) in Fourier space and g is a gravitational acceleration. It is well known that third order terms correspond to nonresonant three-wave processes and can be excluded by using nonlinear canonical transformation.

Considering waves which propagate in same direction we applied canonical transformation not only to remove cubic nonlinear terms but to simplify drastically fourth order terms in Hamiltonian. Unlike in [2],[3] the four-wave interaction coefficient is chosen as follows:

$$\begin{aligned} \tilde{T}_{k_2 k_3}^{k k_1} &= \frac{(k k_1 k_2 k_3)^{\frac{1}{2}}}{2\pi} \min(k, k_1, k_2, k_3) \theta(k k_1 k_2 k_3) = \\ &= \frac{(k k_1 k_2 k_3)^{\frac{1}{2}}}{8\pi} (k + k_1 + k_2 + k_3 - |k - k_2| - |k - k_3| - |k_1 - k_2| - |k_1 - k_3|) \theta(k k_1 k_2 k_3) \end{aligned} \quad (2)$$



Transformation from $b(x, t)$ to physical variables $\eta(x, t)$ and $\psi(x, t)$ up to the second order has the form:

$$\begin{aligned} \eta(x) &= \frac{1}{\sqrt{2}g^{\frac{1}{4}}}(\hat{k}^{-\frac{1}{4}}c(x) + \hat{k}^{-\frac{1}{4}}c(x)^*) + \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{-\frac{1}{4}}c(x) - \hat{k}^{-\frac{1}{4}}c^*(x)]^2, \\ \psi(x) &= -i\frac{g^{\frac{1}{4}}}{\sqrt{2}}(\hat{k}^{-\frac{3}{4}}c(x) - \hat{k}^{-\frac{3}{4}}c(x)^*) + \frac{i}{2}[\hat{k}^{-\frac{1}{4}}c^*(x)\hat{k}^{\frac{1}{4}}c^*(x) - \hat{k}^{-\frac{1}{4}}c(x)\hat{k}^{\frac{1}{4}}c(x)] + \\ &+ \frac{1}{2}\hat{H}[\hat{k}^{-\frac{1}{4}}c(x)\hat{k}^{\frac{1}{4}}c^*(x) + \hat{k}^{-\frac{1}{4}}c^*(x)\hat{k}^{\frac{1}{4}}c(x)] \end{aligned} \quad (3)$$

The third order terms are very cumbersome, so we don't give them here. After the transformation the Hamiltonian take the compact form:

$$H = \int c^* \hat{V} c dx + \frac{1}{2} \int \left[\frac{i}{4} (c^2 \frac{\partial}{\partial x} c^{*2} - c^{*2} \frac{\partial}{\partial x} c^2) - |c|^2 \hat{k} (|c|^2) \right] dx \quad (4)$$

Here operator \hat{V} in k-space is so that $V_k = \frac{\omega_k}{k}$. $\omega_k = \sqrt{gk}$ – dispersion law for surface gravity waves on deep water. Equation of motion is the following:

$$\frac{\partial c(x, t)}{\partial t} + i\hat{\omega}_k c(x, t) - i\hat{P}^+ \frac{\partial}{\partial x} \left(|c(x, t)|^2 \frac{\partial c(x, t)}{\partial x} \right) = \hat{P}^+ \frac{\partial}{\partial x} \left(\hat{k} |c(x, t)|^2 c(x, t) \right) \quad (5)$$

here \hat{P}^+ is projection operator to the upper half-plane.

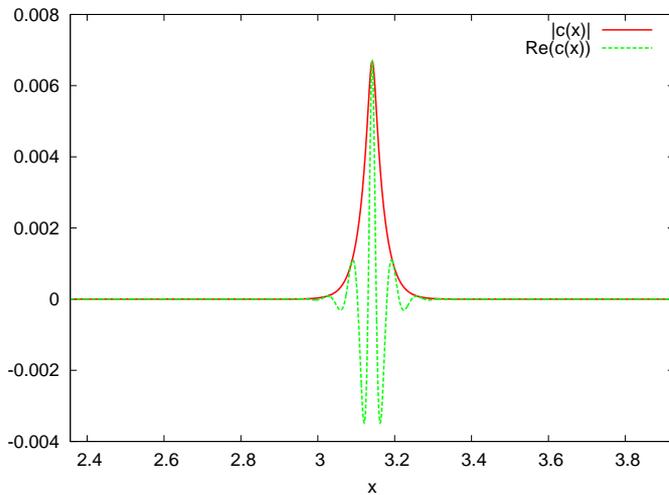


Figure 1. Narrow breather with $V = 5 \cdot 10^{-2}$, $\Omega = 5.2$ (wavenumber of the carrier wave $k_0 \sim 100$). Green dashed curve corresponds to the real part ($Re(c(x, 0))$); red solid curve – modulus ($|c(x, 0)|$)

A monochromatic wave

$$c(x, t) = C_0 e^{i(k_0 x - \omega_0 t)} \quad (6)$$

is the simplest solution of (5). Here C_0 is arbitrary complex constant. Substituting (6) into (5) yields the following relation for frequency shift:

$$\omega_0 = \omega_{k_0} + k_0^2 |C_0|^2. \quad (7)$$

Recalling the transformation (3), one can see that relation (7) coincides with the well-known Stokes correction to the frequency due to finite wave amplitude:

$$\omega_0 = \omega_{k_0} (1 + \frac{1}{2} k_0^2 |\eta_0|^2), \quad |C_0|^2 = \frac{1}{2} \omega_{k_0} \eta_0^2 \quad (8)$$

2. Breathers

Breather is the localized solution (5) following type:

$$c(x, t) = C(x - Vt)e^{i(k_0x - \omega_0t)} \quad (9)$$

Here k_0 is the wavenumber of the carrier wave. In the Fourier space breather can be written as follow:

$$c_k = e^{i(\Omega + Vk)t} \phi_k \quad (10)$$

Such solutions are completely determined by two real parameters: V – group velocity and Ω . Functions ϕ_k satisfies the equation:

$$(\Omega + Vk - \omega_k)\phi_k = \frac{1}{2} \int \tilde{T}_{kk_1}^{k_2k_3} \phi_{k_1}^* \phi_{k_2} \phi_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 \quad (11)$$

Breather (9) can be found by Petviashvili method

$$\begin{aligned} \phi_k^{n+1} &= \frac{NL_k^n}{M_k} \left[\frac{\langle \phi^n \cdot NL(\phi^n) \rangle}{\phi^n \cdot M\phi^n} \right]^\gamma, \quad M_k = \Omega + Vk - \omega_k, \\ NL(\phi^n) &= -P^+ \frac{\partial}{\partial x} \left(|\phi^n|^2 \frac{\partial \phi^n}{\partial x} \right) + iP^+ \frac{\partial}{\partial x} \left(\hat{k} (|\phi^n|^2) \phi^n \right) \end{aligned}$$

Breather solution of this equation in the periodic domain 2π with $V = 5 \cdot 10^{-2}$, $\Omega = 5.2$ (wavenumber of the carrier wave $k_0 \sim 100$) is shown in Fig.1. Figure 2 shows the spectrum of the breather in log scale. These breathers can be considered as a simple model of the freak waves.

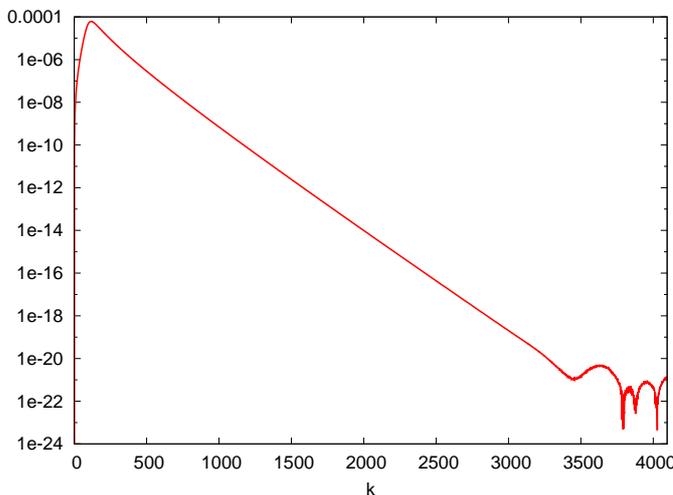


Figure 2. Spectrum ($|c_k(0)|$) of narrow breather with $V = 5 \cdot 10^{-2}$, $\Omega = 5.2$ (wavenumber of the carrier wave $k_0 \sim 100$) in logarithmic scale.

3. Interaction of breathers

Breather is very stable structure. We performed numerical simulation of breathers collision to prove this. The pseudospectral Fourier algorithm was applied for simulation the equation (5). Numerical integration of the equation (5) were carried out on the base of Runge-Kutta method 4th order accurate in time. Simulation was performed in the center-of-momentum frame.

- As the initial condition we have used two breathers separated in space (distance was equal to π .)

- The first breather has the following parameters: $V_1 = 5 \cdot 10^{-2}$, $\Omega_1 = 5.005$ (wavenumber of the carrier wave $k_0 \sim 100$)
- For the second breather $V_2 \simeq 3.535 \cdot 10^{-2}$, $\Omega_2 \simeq 7.081$ (wavenumber of the carrier wave $k_0 \sim 200$)

Collision of two breathers is shown in Fig.3. Fourier spectrum ($|c_k|$) in log scale for different time moments is shown in Fig. 4. One can see that spectrum returns to it's initial state and the breathers profiles almost have not changed.

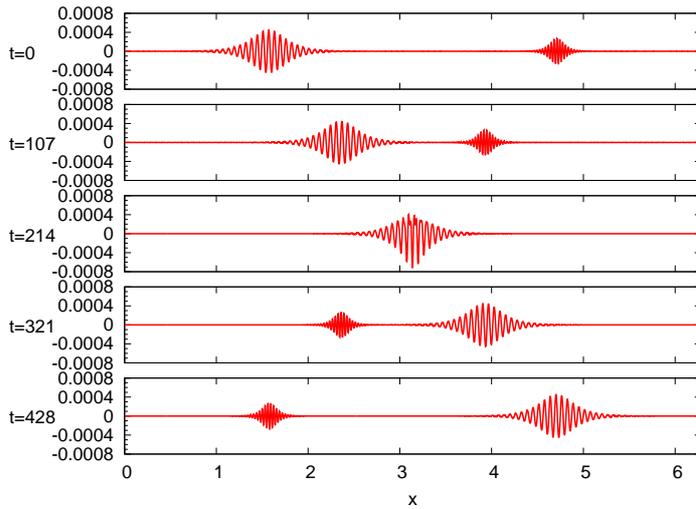


Figure 3. Snapshots of breather collision. Free surface for different time moments $t = 0, 107, 214, 321, 428$.

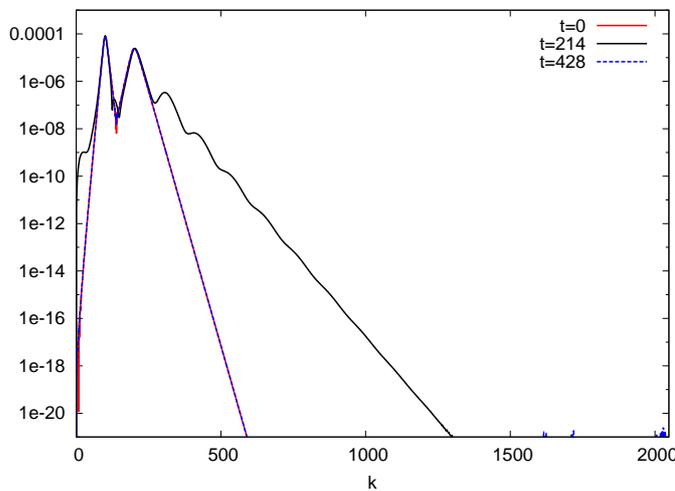


Figure 4. Fourier spectrum ($|c_k|$) in log scale for different time moments $t = 0, 214, 428$. Red solid curve corresponds to initial statement ($t = 0$); black solid curve corresponds to the collision moment ($t = 214$); blue dashed curve – spectrum after breathers collision ($t = 428$)

4. Modulational Instability of Monochromatic wave

The numerical simulation of the modulational instability of the homogeneous wave train in the framework of compact equation (5) was performed by pseudospectral Fourier method. Initial condition was chosen as slightly perturbed monochromatic wave with the steepness $\mu \sim 0.82$ and wavenumber $k_0 = 100$. After ~ 1000 wave periods on a free surface the freak wave was formed. The lifetime of this wave was about 10 periods, after which it disappeared. Snapshots of free surface for different times ($t = 0, 615, 635, 655$) is shown on figure 5. Figure 6 shows the zoomed freak wave profile at $t = 635$. One can see that the freak wave's height is more than 2 times greater than the height of neighbour waves.

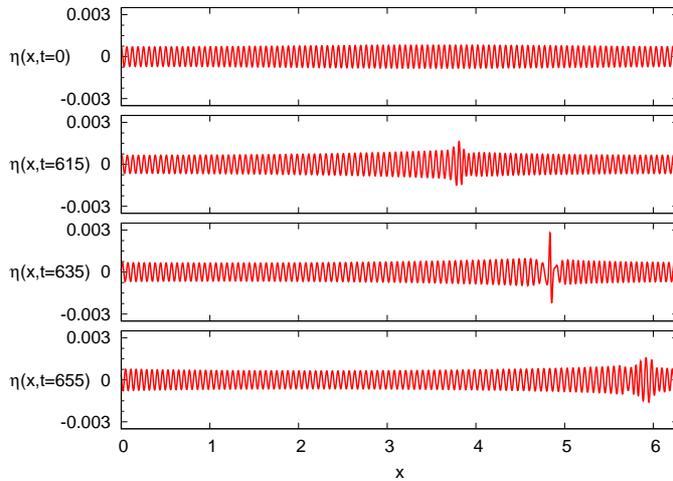


Figure 5. Freak wave formation. Snapshots of free surface for different times ($t = 0, 615, 635, 655$) is shown

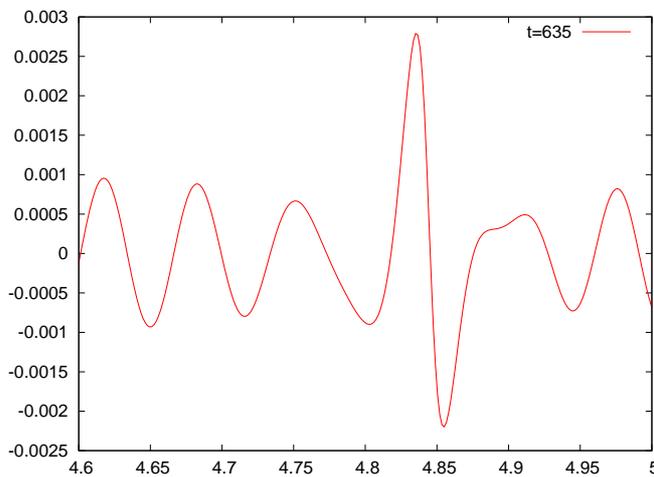


Figure 6. Freak wave on the free surface at $t = 635$.

5. Conclusion

We have performed numerical simulations of nonlinear stage of modulational instability up to the freak-wave formation in the framework of equation (5). The new compact equation (5) can be generalized for the almost 2D waves i.e. waves slightly inhomogeneous in the transverse direction y . In this case frequency $\hat{\omega}$ depends on both k_x and k_y as $\hat{\omega}_{k_x, k_y}$, while nonlinear terms not depend on y , and c now depends on both x and y :

$$\frac{\partial c(x, y, t)}{\partial t} + i\hat{\omega}_{k_x, k_y}c(x, y, t) - i\hat{P}_x^+ \frac{\partial}{\partial x} \left(|c(x, y, t)|^2 \frac{\partial c(x, y, t)}{\partial x} \right) = \hat{P}_x^+ \frac{\partial}{\partial x} \left(\hat{k}_x |c(x, y, t)|^2 c(x, y, t) \right) \quad (12)$$

Due to specific structure of nonlinearity the equation (12) can be effectively solved numerically.

Acknowledgments

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References

- [1] Zakharov V E 1968 *J. Appl. Mech. Tech. Phys* **9** 190–4
- [2] Dyachenko A I and Zakharov V E 2011 *JETP Lett.* **93(12)** 701–5
- [3] Dyachenko A I and Zakharov V E 2012 *Europ. J. Mech. B* **32** 17–21